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Linear vs. nonlinear problems - Part II

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Nonlinear solution methods

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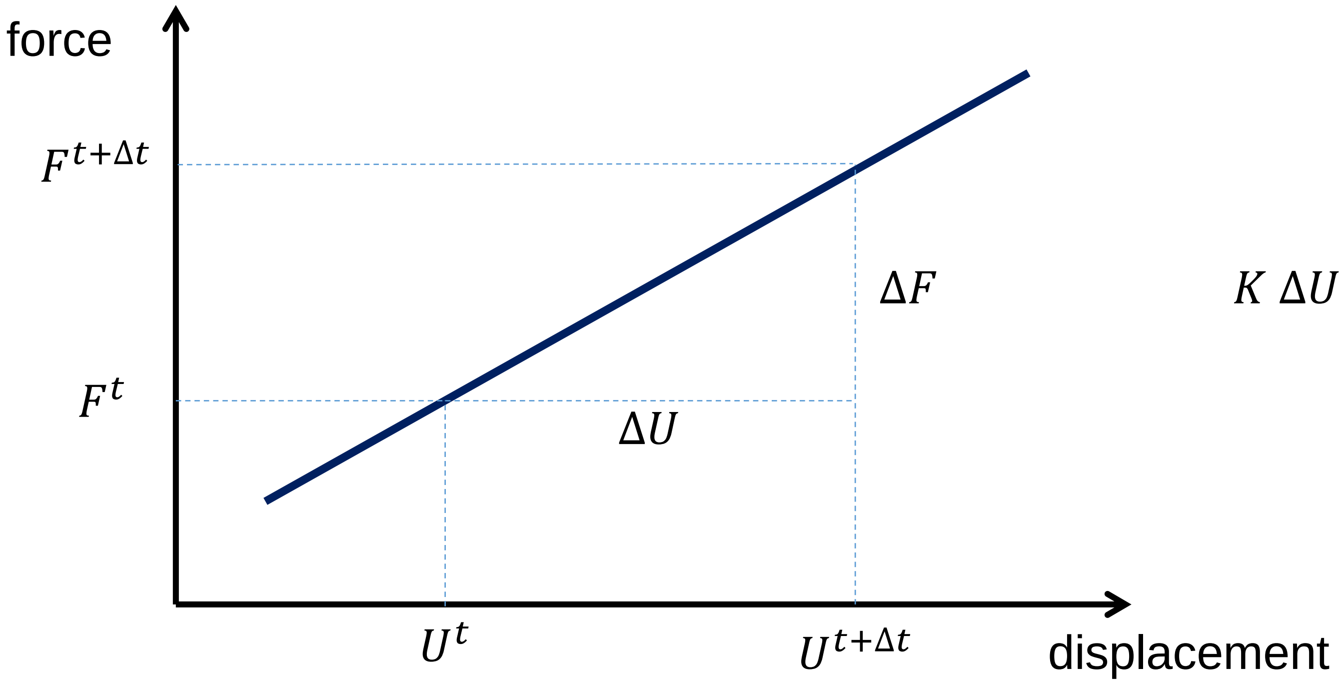
Linear: $K\ U = F \ \rightarrow \begin{bmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{bmatrix} \begin{Bmatrix} \vdots \end{Bmatrix} = \begin{Bmatrix} \vdots \end{Bmatrix}$, $K = \text{const.}$



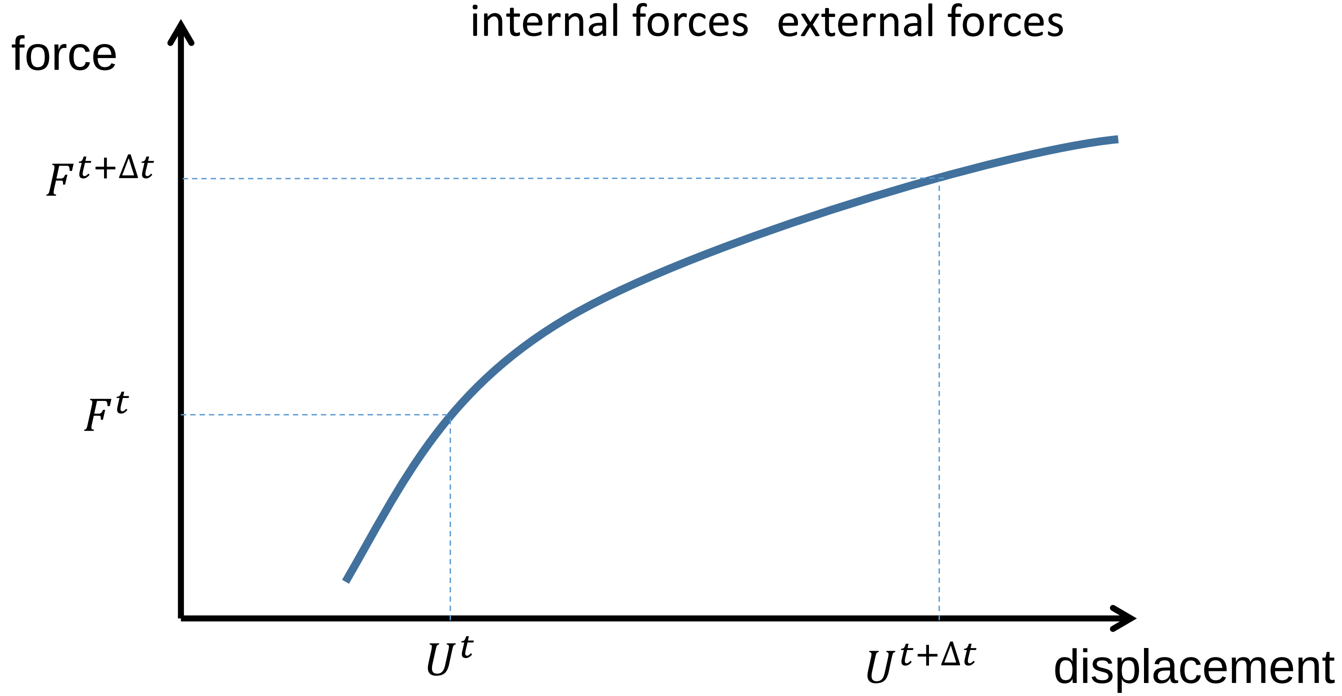
$K\ \Delta U = \Delta F$

Nonlinear solution methods

Nonlinear: $K(U)\,U = F \;\rightarrow\; \underbrace{F_{int}(U)}_{\text{internal forces}} = \underbrace{F_{ext}}_{\text{external forces}}$

$$\underbrace{F_{int}(U) - F_{ext}}_{} = 0$$

force residual: $R(U) = 0$

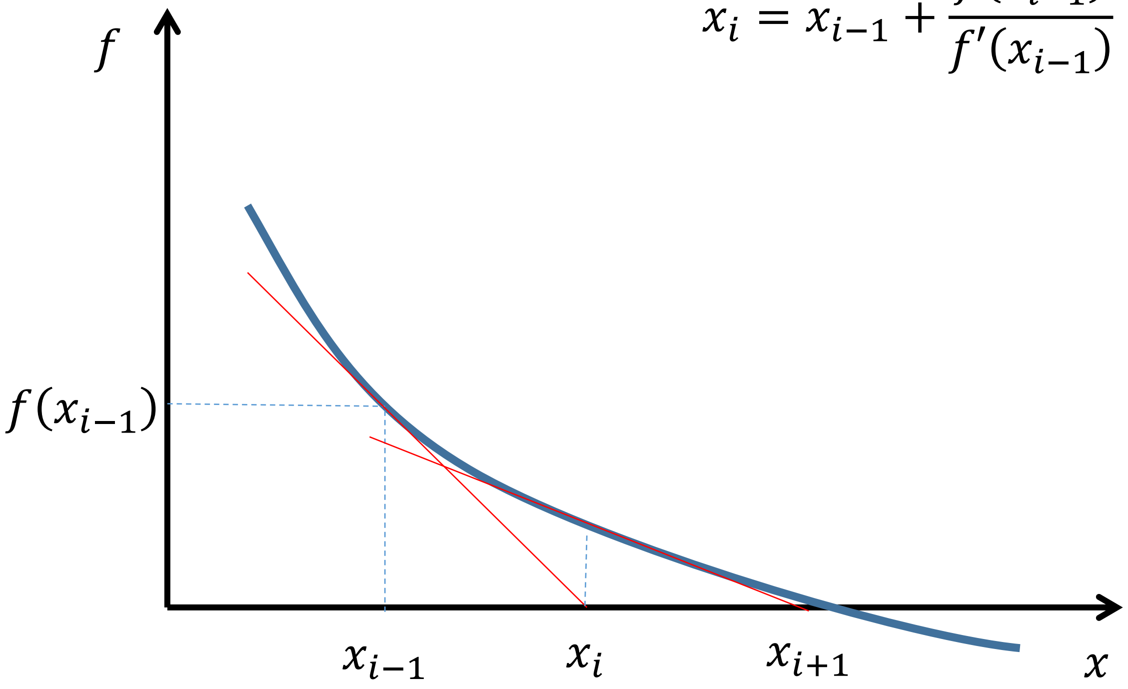


Nonlinear solution methods

Newton-Raphson method

Taylor series expansion: $\cancel{f(x_i)} = f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1}) + \cancel{\text{higher order terms}}$

$$x_i = x_{i-1} + \frac{f(x_{i-1})}{f'(x_{i-1})}$$



Nonlinear solution methods

Newton-Raphson method

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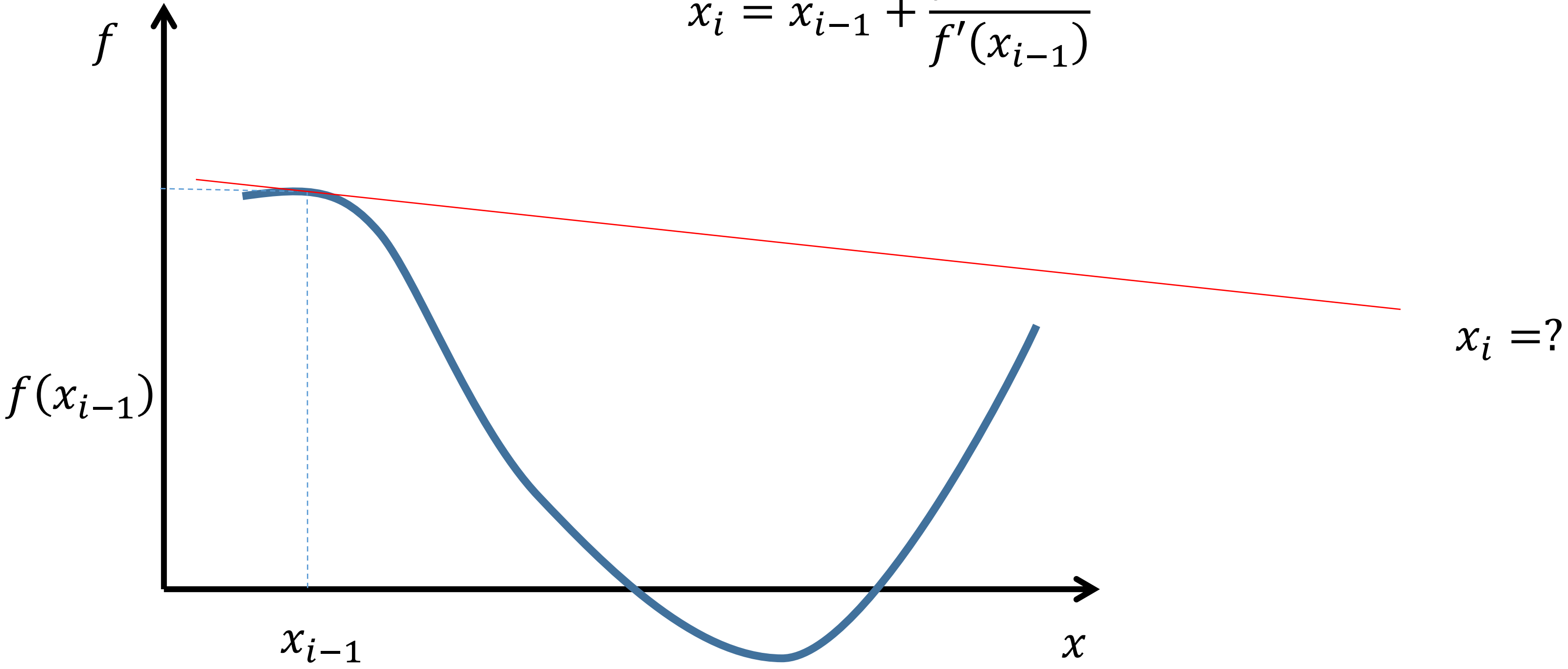
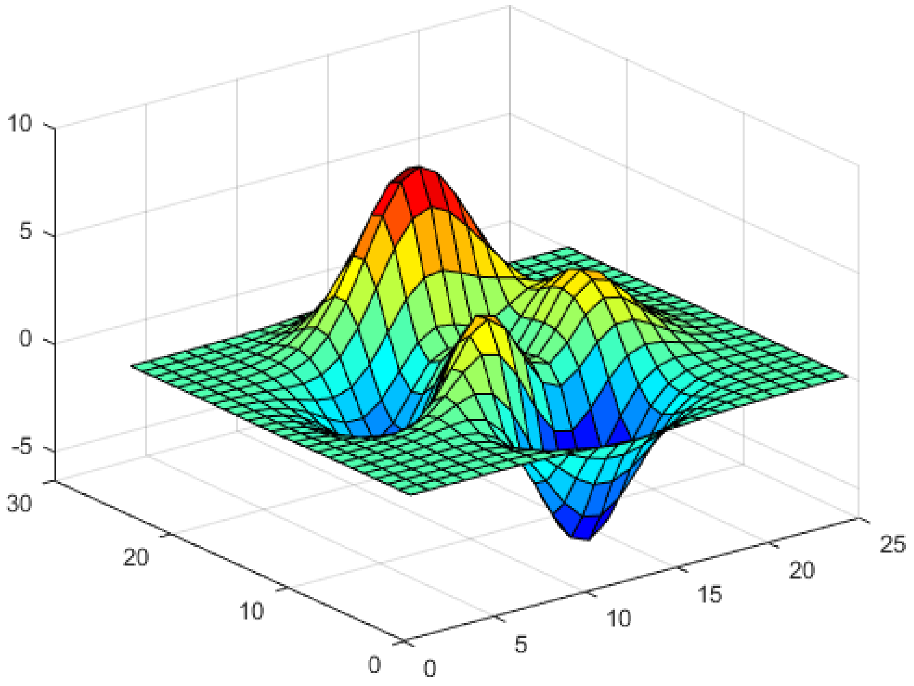
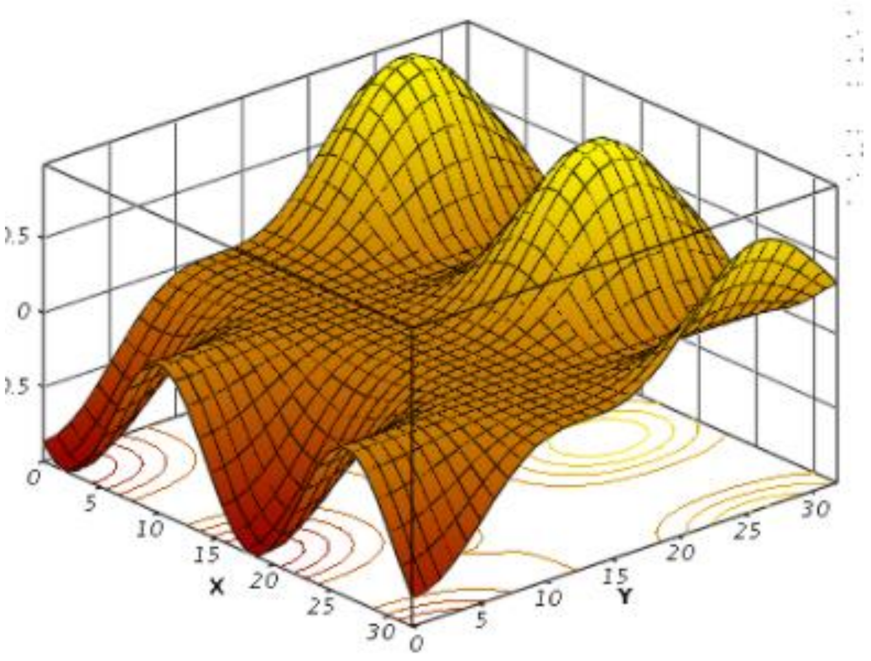


Newton-Raphson method





[1]

[1] https://skill-lync.com/student-projects/genetic-algorithm-and-global-maxima-of-a-function-in-matlab-octave

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Newton-Raphson method - multiple degrees of freedom

$$\mathbf{R}(\mathbf{U}) = \mathbf{F}_{int}(\mathbf{U}) - \mathbf{F}_{\mathbf{ext}} = \mathbf{0} \quad , \qquad \mathbf{U} = ?$$

$$\mathbf{R}\left(\mathbf{U}^{t+\Delta t}\right) = \mathbf{F}_{int}^{t+\Delta t}\left(\mathbf{U}^{t+\Delta t}\right) - \mathbf{F}_{ext}^{t+\Delta t} = \mathbf{0}$$

$$\mathbf{R}\left(\mathbf{U}^{t+\Delta t,i}\right) = \mathbf{R}\left(\mathbf{U}^{t+\Delta t,i-1}\right) + \frac{\partial \mathbf{R}}{\partial \mathbf{U}}\left(\mathbf{U}^{t+\Delta t,i} - \mathbf{U}^{t+\Delta t,i-1}\right) + \text{higher order terms}$$

$$\mathbf{0} = \mathbf{R}\left(\mathbf{U}^{t+\Delta t,i-1}\right) + \underbrace{\frac{\partial \mathbf{R}}{\partial \mathbf{U}}}_{\text{tangent stiffness matrix}}\boldsymbol{\delta}\mathbf{U}^{i} \longrightarrow \left(\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}} - \cancel{\frac{\partial \mathbf{F}_{ext}}{\partial \mathbf{U}}}\right)\boldsymbol{\delta}\mathbf{U}^{i} = \mathbf{F}_{ext}^{t+\Delta t} - \mathbf{F}_{int}^{t+\Delta t,i-1}\left(\mathbf{U}^{t+\Delta t,i-1}\right)$$

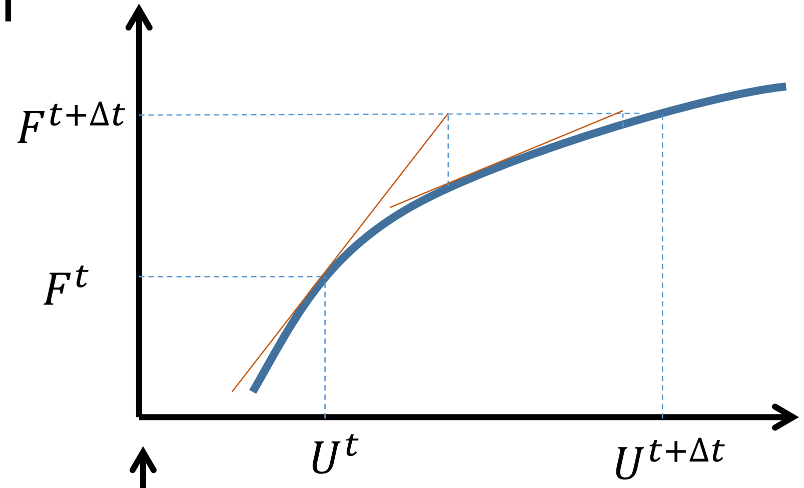
$$\boldsymbol{\delta}\mathbf{U}^{i} = \ldots$$

Nonlinear solution methods

Newton-Raphson method - multiple degrees of freedom

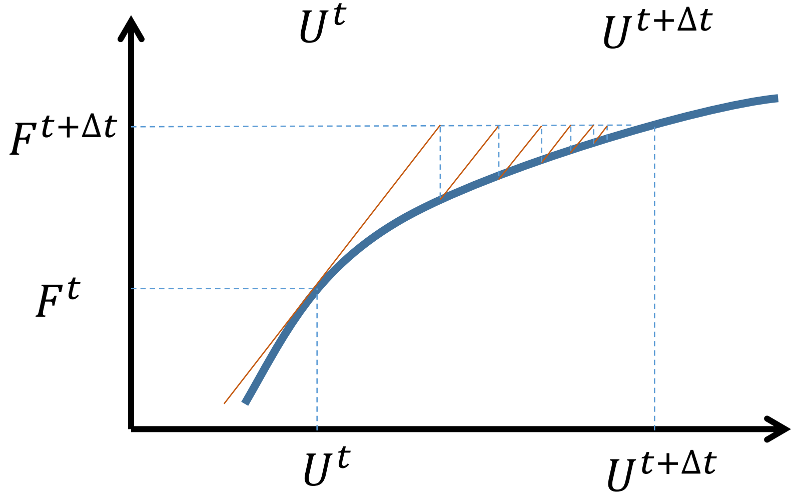
$$\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}} \delta \mathbf{U}^{i} = \mathbf{F}_{ext}^{t+\Delta t} - \mathbf{F}_{int}^{t+\Delta t,i-1}\left(\mathbf{U}^{t+\Delta t,i-1}\right)$$

$\left.\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}}\right|_{t+\Delta t,i-1}$, full Newton-Raphson method



$\left.\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}}\right|_{0}$, initial stress method

$\left.\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}}\right|_{t}$, modified Newton-Raphson method



$\left.\frac{\partial \mathbf{F}_{int}}{\partial \mathbf{U}}\right|_{?}$, update at certain times only

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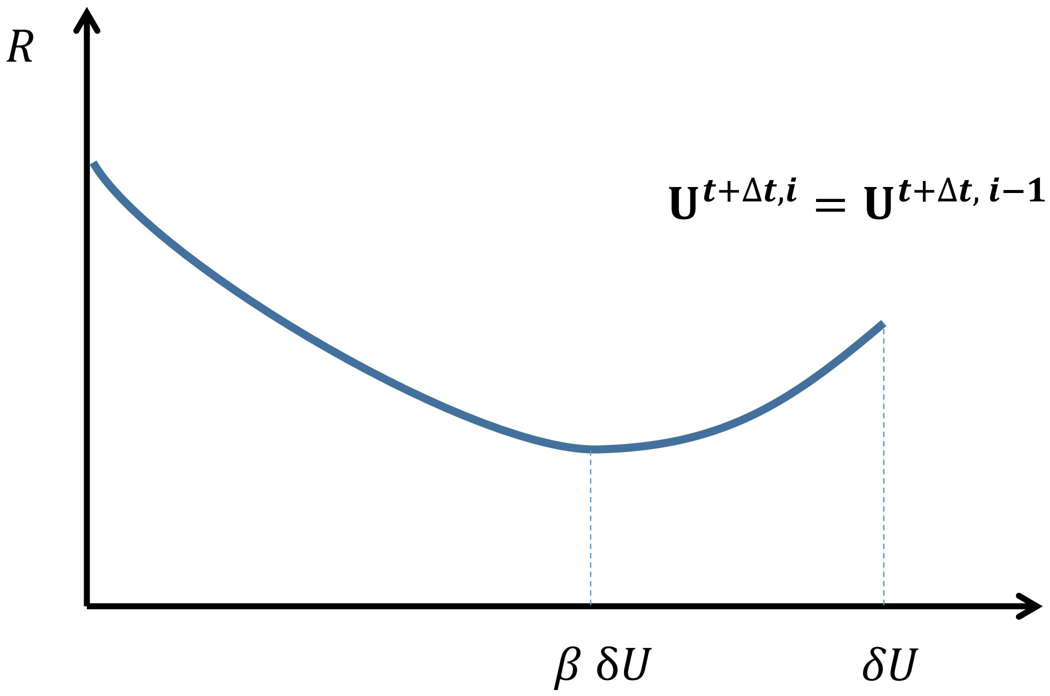
Line search

$$\mathbf{U}^{t+\Delta t,i} = \mathbf{U}^{t+\Delta t,\,i-1} + \boldsymbol{\delta}\mathbf{U}^{i}$$



$$\mathbf{U}^{t+\Delta t,i} = \mathbf{U}^{t+\Delta t,\,i-1} + \beta\,\boldsymbol{\delta}\mathbf{U}^{i}$$

Nonlinear solution methods

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Convergence criteria

- energy
$$\frac{\boldsymbol{\delta}\mathbf{U}^{\boldsymbol{i}}\left(\mathbf{F}_{ext}^{t+\Delta t}-\mathbf{F}_{int}^{t+\Delta t,i-1}\right)}{\boldsymbol{\delta}\mathbf{U}^{\mathbf{1}}\left(\mathbf{F}_{ext}^{t+\Delta t}-\mathbf{F}_{int}^{t}\right)}<\epsilon$$

- force residuals
$$\frac{\left\|\mathbf{F}_{ext}^{t+\Delta t}-\mathbf{F}_{int}^{t+\Delta t,i-1}\right\|}{\mathrm{RNORM}}<\epsilon$$

- displacement
$$\frac{\left\|\boldsymbol{\delta}\mathbf{U}^{\boldsymbol{i}}\right\|}{\mathrm{DNORM}}<\epsilon$$

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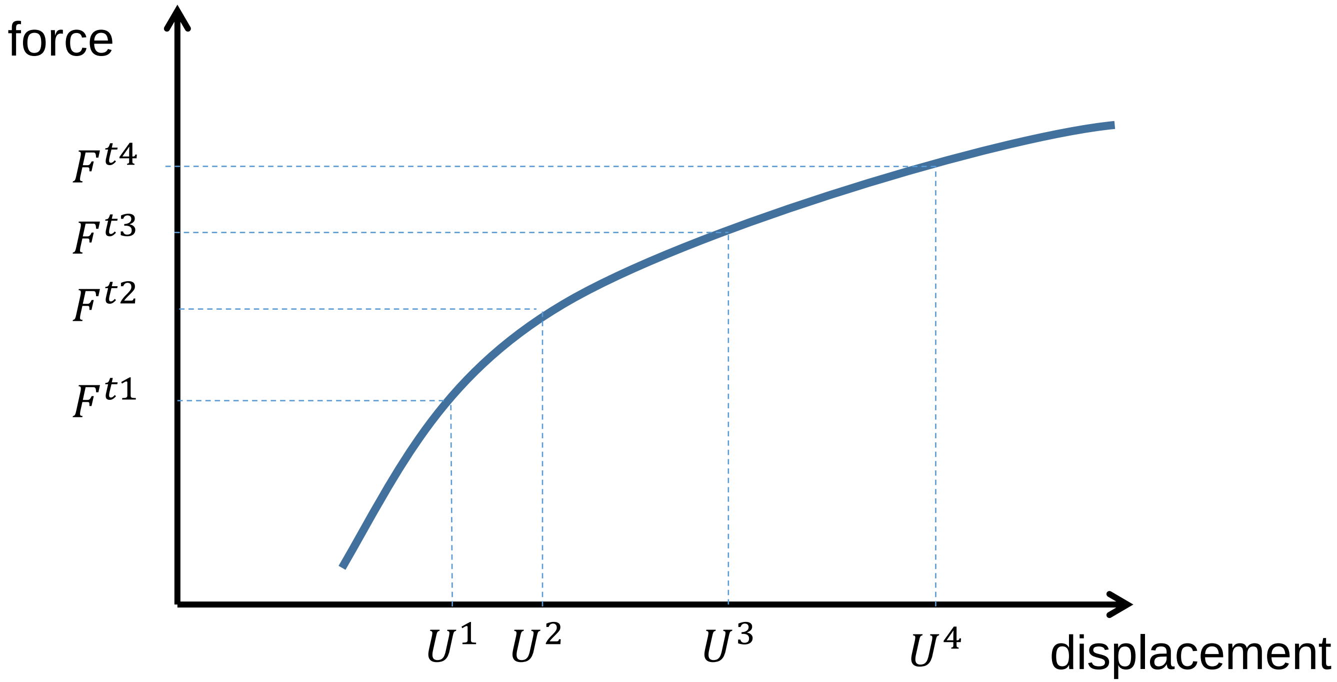
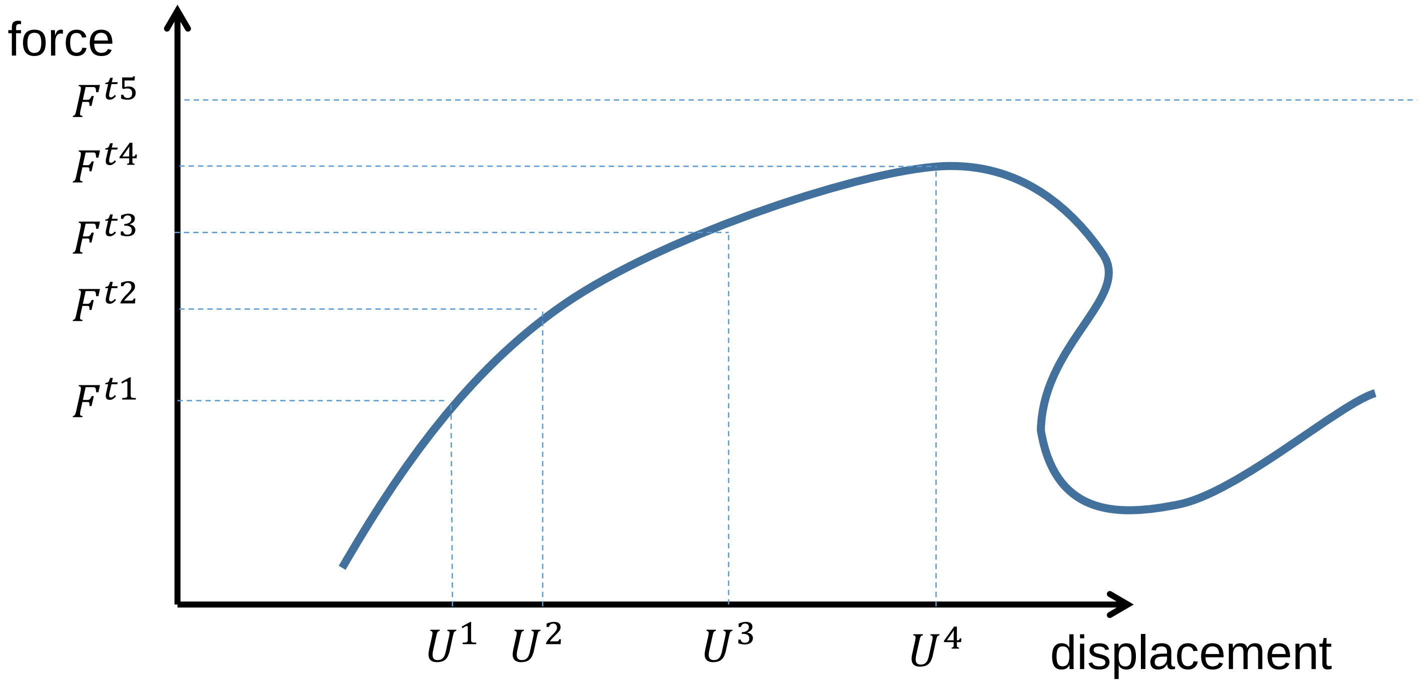
Incrementation



Arc-length method



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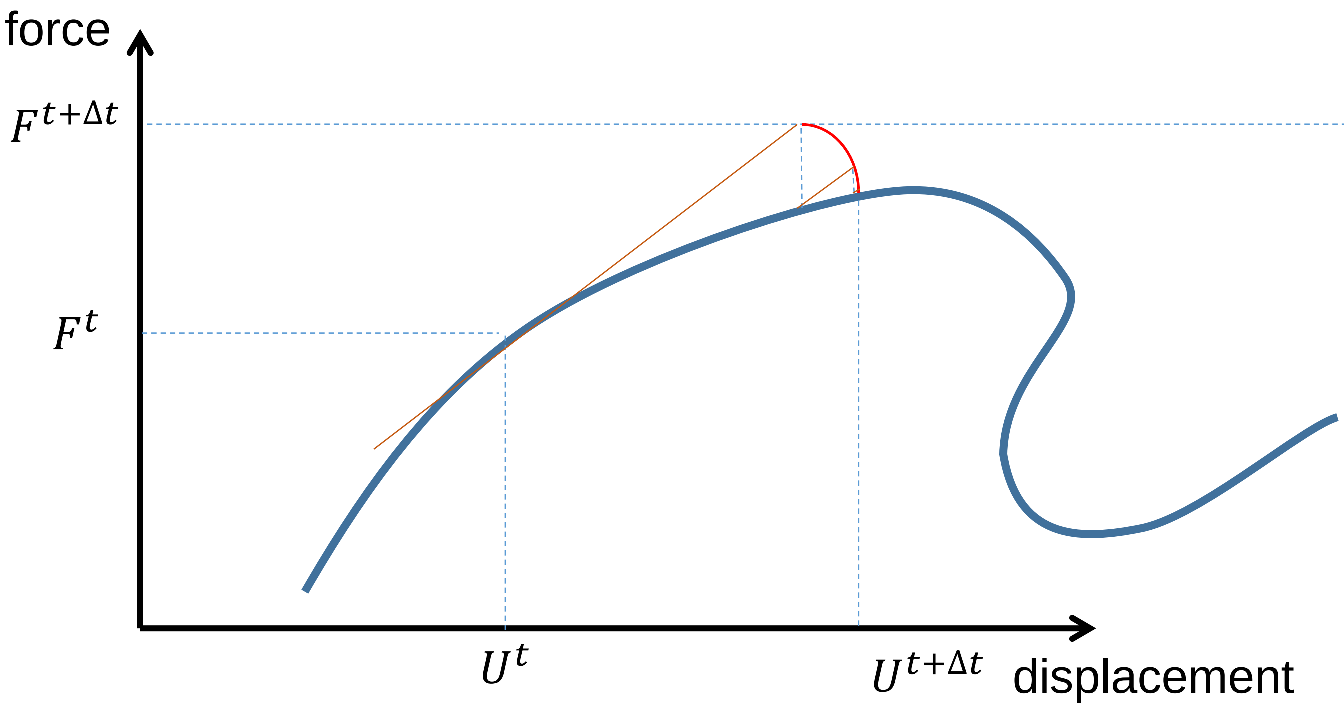
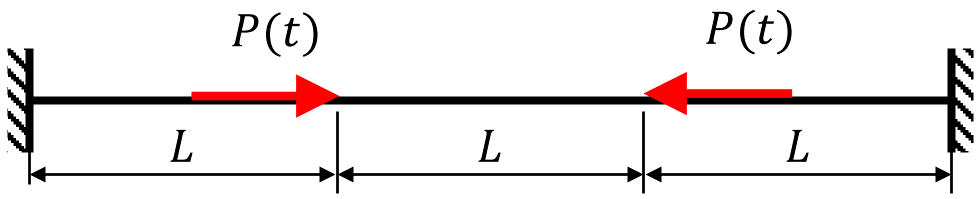
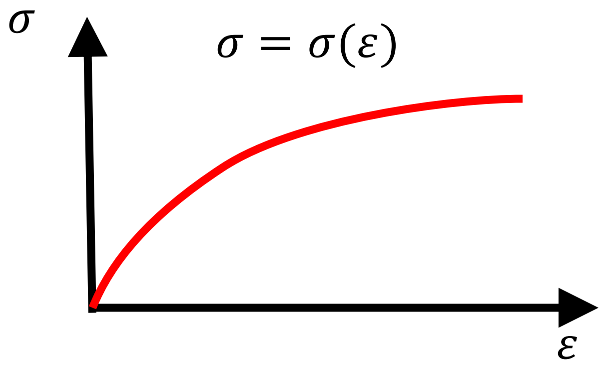
Arc-length method

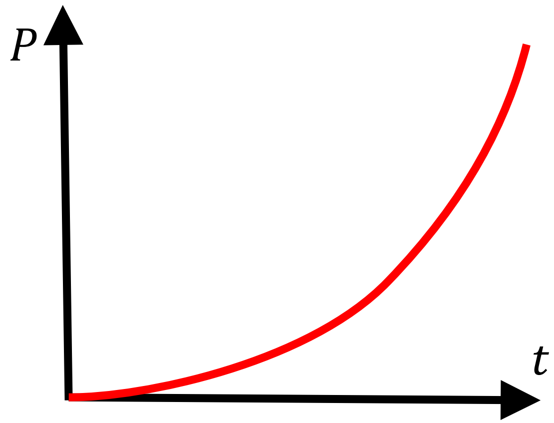


Nonlinear solution methods

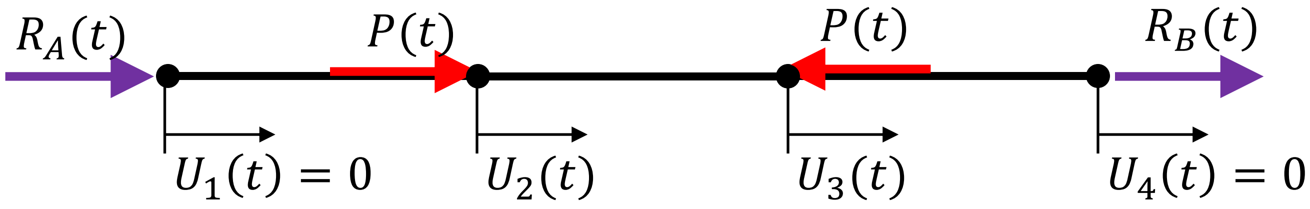
Example: material nonlinearity







FEM:



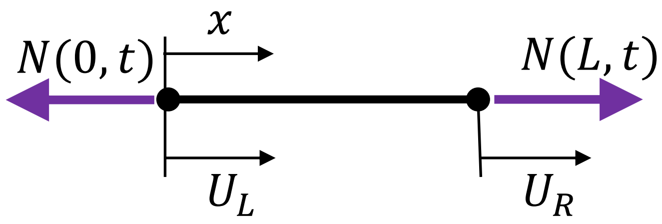
Example: material nonlinearity

$$N' = -n$$

$$N' = 0$$

$$(\sigma(x,t)A)' = 0$$

$$\int_0^L \sigma'(x,t)A\, v(x)\,\mathrm{d}x = 0$$



$$(\sigma(x,t)A\, v(x))\Big|_0^L - \int_0^L \sigma(x,t)A\, v'(x,t)\,\mathrm{d}x = 0$$

$$\int_0^L \sigma(x,t)A\, v'(x,t)\,\mathrm{d}x = N(L,t)\, v(L) - N(0,t)\, v(L)$$

... weak form

Nonlinear solution methods

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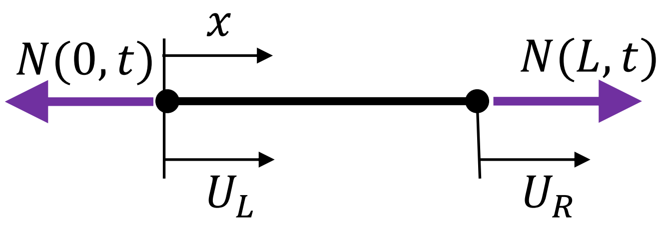
Example: material nonlinearity



$$\int_0^L \sigma(x,t) A\, v'(x,t)\, \mathrm{d}x = N(L,t)\, v(L) - N(0,t)\, v(L)$$ ... weak form

$$u(x,t) = U_L \left(1 - \frac{x}{L}\right) + U_R\, \frac{x}{L}$$ ... approximation of displacement field

$$\varepsilon(x,t) = u'(x,t) = \frac{1}{L}(U_R - U_L) \longrightarrow \sigma(x,t) = E\,\varepsilon(x,t) = \frac{E}{L}(U_R - U_L)$$ ... linear

$$\sigma(x,t) = f(\varepsilon(x,t))$$ ... nonlinear

Galerkin:

$$v(x,t) = 1 - \frac{x}{L} \quad \to \quad -\int_0^L \sigma(x,t)\, A\, \frac{1}{L}\, \mathrm{d}x = -N(0,t) \quad \to \quad -\sigma(t)\, A = -N(0,t)$$

$$v(x,t) = \frac{x}{L} \quad \to \quad \int_0^L \sigma(x,t)\, A\, \frac{1}{L}\, \mathrm{d}x = N(L,t) \quad \to \quad \sigma(t)\, A = N(L,t)$$

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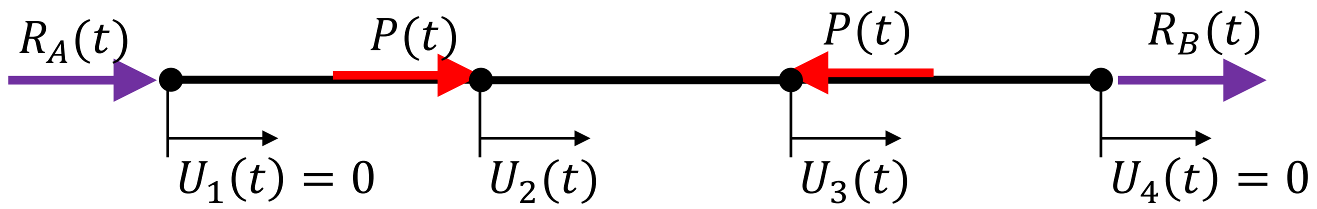
Example: material nonlinearity

Assembling:



$$\underbrace{\begin{Bmatrix} -\sigma_1\,(t)\,A \\ \sigma_1\,(t)\,A - \sigma_2\,(t)\,A \\ \sigma_2\,(t)\,A - \sigma_3\,(t)\,A \\ \sigma_3\,(t)\,A \end{Bmatrix}}_{\mathbf{F}_{\text{int}}\,(t)} = \underbrace{\begin{Bmatrix} -R_A(t) \\ P(t) \\ -P(t) \\ R_B(t) \end{Bmatrix}}_{\mathbf{F}_{\text{ext}}(t)}$$

Nonlinear solution methods

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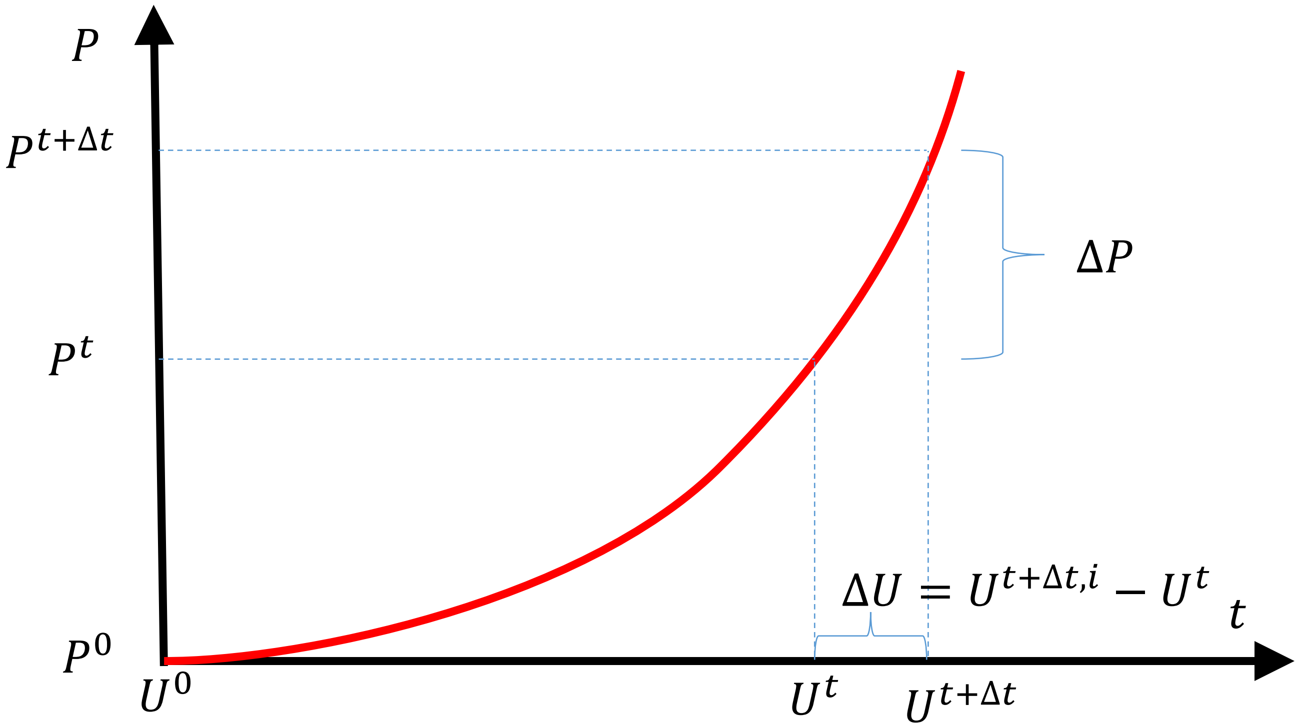
Example: material nonlinearity

Time discretization:



$\mathbf{F}_{\text{int}}^{t}(\mathbf{U}^t) - \mathbf{F}_{\text{ext}}^{t} = \mathbf{0}$ ... fulfilled

$\mathbf{F}_{\text{int}}^{t+\Delta t}\left(\mathbf{U}^{t+\Delta t}\right) - \mathbf{F}_{\text{ext}}^{t+\Delta t} = \mathbf{0}$ ... requested!

$$\frac{\partial \mathbf{F}_{\text{int}}}{\partial \mathbf{U}} \boldsymbol{\delta}\mathbf{U}^{i} = \mathbf{F}_{\text{ext}}^{t+\Delta t} - \mathbf{F}_{\text{int}}^{t+\Delta t,i-1}\left(\mathbf{U}^{t+\Delta t,i-1}\right)$$

$$\mathbf{U}^{t+\Delta t,i} = \mathbf{U}^{t+\Delta t,i-1} + \boldsymbol{\delta}\mathbf{U}^{i}$$

Example: material nonlinearity

Iterations:

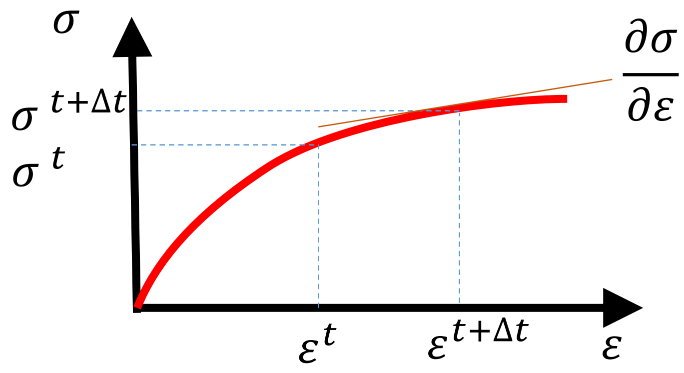
$$\frac{\partial \mathbf{F}_{\text{int}}}{\partial \mathbf{U}} \underbrace{\boldsymbol{\delta}\mathbf{U}^{i}}_{\text{unknown}} = \underbrace{\mathbf{F}_{\text{ext}}^{t+\Delta t} - \mathbf{F}_{\text{int}}^{t+\Delta t,i-1}\left(\mathbf{U}^{t+\Delta t,i-1}\right)}_{\text{known}}$$

$$\mathbf{F}_{\text{int}}(\mathbf{U}) = \mathbf{F}_{\text{int}}\left(\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}(\mathbf{U})\right)\right)$$

$$\frac{\partial \mathbf{F}_{\text{int}}}{\partial \mathbf{U}} = \frac{\partial \mathbf{F}_{\text{int}}}{\partial \boldsymbol{\sigma}}\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{U}}$$

$$\underbrace{\begin{Bmatrix} -\sigma_1(t)\,A \\ \sigma_1(t)\,A - \sigma_2(t)\,A \\ \sigma_2(t)\,A - \sigma_3(t)\,A \\ \sigma_3(t)\,A \end{Bmatrix}}_{\mathbf{F}_{\text{int}}(t)} = \underbrace{\begin{Bmatrix} -R_A(t) \\ P(t) \\ -P(t) \\ R_B(t) \end{Bmatrix}}_{\mathbf{F}_{\text{ext}}(t)} \quad \rightarrow \quad \frac{\partial \mathbf{F}_{\text{int}}}{\partial \boldsymbol{\sigma}}$$

$$\varepsilon(x,t) = u'(x,t) = \frac{1}{L}(U_R - U_L) \quad \rightarrow \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \frac{1}{L}(U_2 - U_1) \\ \frac{1}{L}(U_3 - U_2) \\ \frac{1}{L}(U_4 - U_3) \end{Bmatrix} \rightarrow \quad \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{U}}$$



$$\rightarrow \quad \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$$

Nonlinear solution methods

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Example: material nonlinearity

$$
\begin{bmatrix}
\frac{A\frac{\partial\sigma_1}{\partial\varepsilon_1}}{L} & -\frac{A\frac{\partial\sigma_1}{\partial\varepsilon_1}}{L} & 0 & 0 \\
-\frac{A\frac{\partial\sigma_1}{\partial\varepsilon_1}}{L} & \frac{A\left(\frac{\partial\sigma_1}{\partial\varepsilon_1}+\frac{\partial\sigma_2}{\partial\varepsilon_2}\right)}{L} & -\frac{A\frac{\partial\sigma_2}{\partial\varepsilon_2}}{L} & 0 \\
 & & \frac{A\left(\frac{\partial\sigma_2}{\partial\varepsilon_2}+\frac{\partial\sigma_3}{\partial\varepsilon_3}\right)}{L} & -\frac{A\frac{\partial\sigma_3}{\partial\varepsilon_3}}{L} \\
 & \text{symmetric} & & \frac{A\frac{\partial\sigma_3}{\partial\varepsilon_3}}{L}
\end{bmatrix}
\begin{Bmatrix} \delta U_1 \\ \delta U_2 \\ \delta U_3 \\ \delta U_4 \end{Bmatrix}
=
\begin{Bmatrix} F_{\text{ext1}} - F_{\text{int1}} \\ F_{\text{ext2}} - F_{\text{int2}} \\ F_{\text{ext3}} - F_{\text{int3}} \\ F_{\text{ext4}} - F_{\text{int4}} \end{Bmatrix}
$$

Nonlinear solution methods

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Example: material nonlinearity

$\boldsymbol{\delta \mathbf{U}^{i}}$

Update: $\mathbf{U}^{t+\Delta t,i} = \mathbf{U}^{t+\Delta t,\,i-1} + \boldsymbol{\delta \mathbf{U}^{i}}$

$\boldsymbol{\varepsilon}$

$\boldsymbol{\sigma}$

$\mathbf{F}_{\mathrm{int}}$

NEW ITERATION

Convergence check:

NO

$\|\mathbf{F}_{\mathrm{int}} - \mathbf{F}_{\mathrm{ext}}\| < \mathbf{toler}$

YES

NEW INCREMENT

Nonlinear solution methods

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Summary:

- three types of nonlinearity (geometric, material, contact)
- decide, if any of it important for your problem
- incremental loading, iterative process -> lenghty calculations
- choose the appropriate solving method

Take advantage of computational simplicity of linear analyses when possible...

... but do not analyse nonlinear problems with linear analyses!

Thank you for your attention!

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