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Introduction to the Navier-Stokes Equations

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CINECA

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Conservative laws

- Continuum hypothesis,
- Lagrangian vs. Eulerian,
- Reynolds transport theorem,
- **Conservation laws** for fluids,
- Source terms for NS equation
- Navier Stokes equations

Continuum Hypothesis

Continuum hypothesis

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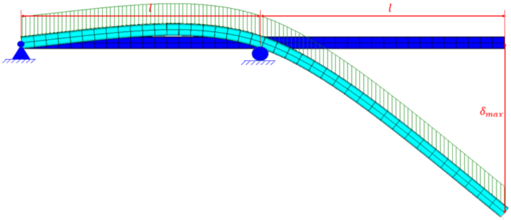
- Navier Stokes equations describe the behaviour of matter on a macroscopic scale which is large with respect to the distance between molecules whose structure does not need to be taken into account explicitly,
- the main idea is to consider a control volume (material element) that has to be:
  - big enough to avoid the description of the molecules,
  - small enough to consider valid the differential calculus

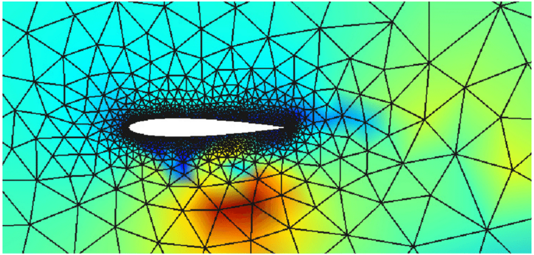
- Material Element *From the observational viewpoint, the reason why the particle structure of the fluid is irrelevant is that the sensitive volume of a certain instrument embedded in the fluid itself is small enough for the measurement to be a local one relative to the macroscopic scale even if it is large enough for the fluctuations arising from the molecular motion to have no effect on the observed average. If the volume of fluid to which the instrument responds were comparable with the volume in which variations due to molecular fluctuations take place, observations would fluctuate from one observation to another and the results would vary in an irregular way with the size of the sensitive volume of the instrument.*

Lagrangian vs. Eulerian

- LAGRANGIAN: typically used in structure modeling - a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time



- EULERIAN: typically used in fluid dynamics - a way of looking at fluid motion where the observer focuses on specific locations in the space (control volume) through which the fluid flows as time passes



- EULERIAN: typically used in fluid dynamics - a way of looking at fluid motion where the observer focuses on specific locations in the space (control volume) through which the fluid flows as time passes

- LAGRANGIAN: typically used in structure modeling - a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time

Reynolds Transport Theorem

- Generally speaking, the Reynolds Transport Theorem is employed to evaluate the rate of change of volume integrals (Φ) of a material property (ϕ) when the volume is changing in time:

$$\Phi(t) = \int_{\Omega(t)} \phi(\mathbf{x}, t) d\Omega, \qquad \mathbf{x} = \xi\left(\mathbf{X}, t\right)$$

Where $\phi\left(\mathbf{x}, \mathbf{t}\right)$ is a function of actual position $(\mathbf{x})$ and time (t). $\mathbf{X}$ is the reference configuration of the system.

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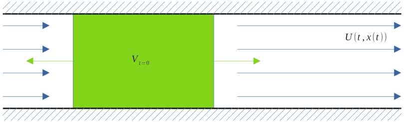

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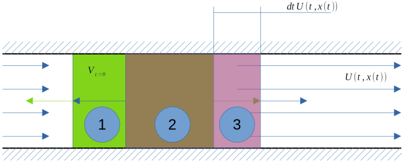

- Generally speaking, the Reynolds Transport Theorem is employed to evaluate the rate of change of volume integrals (Φ) of a material property (ϕ) when the volume is changing in time:

$$\frac{d\Phi}{dt} = \int_{\Omega(t)} \frac{\partial \phi(\mathbf{x}, t)}{\partial t} d\Omega + \oint_{\partial\Omega(t)} \left(\frac{\partial \mathbf{x}}{\partial t} \mathbf{n} \right) \phi(\mathbf{x}, t) d\partial\Omega = 0$$

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$$\frac{d\Phi}{dt} = \int_{\Omega(t)} \frac{\partial \phi(\mathbf{x},t)}{\partial t} d\Omega + \oint_{\partial\Omega(t)} (\mathbf{u}\ \phi(\mathbf{x},t)) \cdot \mathbf{n} d\partial\Omega = 0$$

$$\frac{d\Phi}{dt} = \int_{\Omega(t)} \frac{\partial \phi(\mathbf{x},t)}{\partial t} d\Omega + \int_{\Omega(t)} \frac{\partial\,(\mathbf{u}\ \phi(\mathbf{x},t))}{\partial \mathbf{x}} d\Omega = 0$$

Conservation Laws for Fluids

Conservation laws for fluids

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- Noether's first theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law,
- In mechanics there are conservation laws pertaining to mass M, momentum Q, energy E,
- It is possible to use the Reynolds transport theorem to write the balance equations for a fluid.

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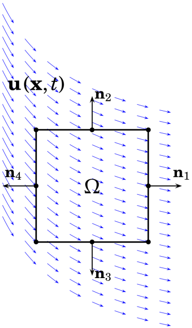
Given a volume of fluid Ω, it is hence possible to write the explicit formulas for the conserved mechanical quantities contained in it.

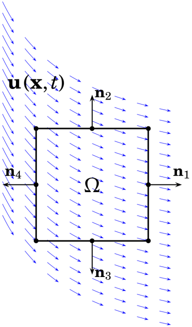


$$M = \int_\Omega \rho \, d\Omega \Rightarrow \text{MASS}$$

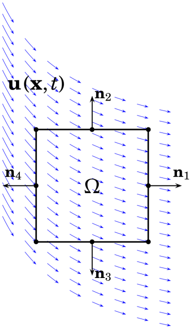
$$Q = \int_\Omega \rho \mathbf{u} \, d\Omega \Rightarrow \text{MOMENTUM}$$

$$E = \int_\Omega \rho e \, d\Omega \Rightarrow \text{ENERGY}$$

Given a volume of fluid Ω, it is hence possible to write the explicit formulas for the conserved mechanical quantities contained in it.

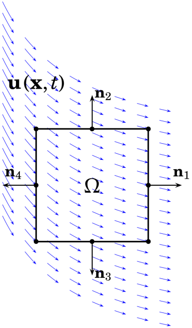


$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \rho \, d\Omega = 0 \Rightarrow \text{MASS} \\ \dfrac{d}{dt}\displaystyle\int_\Omega \rho\mathbf{u} \, d\Omega = \mathbf{0} \Rightarrow \text{MOMENTUM} \\ \dfrac{d}{dt}\displaystyle\int_\Omega \rho e \, d\Omega = 0 \Rightarrow \text{ENERGY} \end{cases}$$



$$
\begin{cases}
\displaystyle\int_\Omega \frac{\partial \rho}{\partial t} d\Omega & + \displaystyle\oint_{\partial\Omega} (\rho)\, u_i n_i \, d\partial\Omega & = & 0 \\
\displaystyle\int_\Omega \frac{\partial \rho u_j}{\partial t} d\Omega & + \displaystyle\oint_{\partial\Omega} (\rho u_j)\, u_i n_i \, d\partial\Omega & = & 0 \\
\displaystyle\int_\Omega \frac{\partial \rho e}{\partial t} d\Omega & + \displaystyle\oint_{\partial\Omega} (\rho e)\, u_i n_i \, d\partial\Omega & = & 0
\end{cases}
$$

Using the
divergence theorem it is possible to rewrite all the equations in therms of volume integrals:



$$\begin{cases} \int_{\Omega} \frac{\partial \rho}{\partial t} d\Omega + \int_{\Omega} \frac{\partial (\rho) u_i}{\partial x_i} d\Omega = 0 \\ \int_{\Omega} \frac{\partial \rho u_j}{\partial t} d\Omega + \int_{\Omega} \frac{\partial (\rho u_j) u_i}{\partial x_i} d\Omega = 0 \\ \int_{\Omega} \frac{\partial \rho e}{\partial t} d\Omega + \int_{\Omega} \frac{\partial (\rho e) u_i}{\partial x_i} d\Omega = 0 \end{cases}$$

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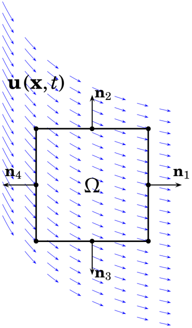
Since no assumption has been done on the volume Ω, the system must be valid for all the possible volumes Ω.



$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho) u_i}{\partial x_i} = 0 \\ \dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial (\rho u_j) u_i}{\partial x_i} = 0 \\ \dfrac{\partial \rho e}{\partial t} + \dfrac{\partial (\rho e) u_i}{\partial x_i} = 0 \end{cases}$$

Conservation laws for fluids

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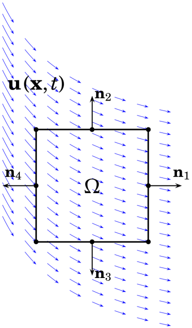

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \nabla\left(\rho \mathbf{u}\right) = 0 \\
\dfrac{\partial \rho u_j}{\partial t} + \nabla\left(\rho u_j \mathbf{u}\right) = 0 \\
\dfrac{\partial \rho e}{\partial t} + \nabla\left(\rho e \mathbf{u}\right) = 0
\end{cases}
$$

$$\mathbf{w} = \left\{ \begin{array}{c} \rho \\ \rho u_{\mathrm{j}} \\ \rho e \end{array} \right\}, \quad \mathbf{F}\left(\mathbf{w}\right) = \left\{ \begin{array}{c} \rho \mathbf{u} \\ \rho \mathbf{u} u_{\mathrm{j}} \\ \rho e \mathbf{u} \end{array} \right\}$$

- differential form:

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \left(\mathbf{F}\left(\mathbf{w}\right)\right) = \mathbf{0}$$

- differential advective form:

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}\left(\mathbf{w}\right) \nabla \left(\mathbf{w}\right) = \mathbf{0}$$

Source Terms for Navier-Stokes equations

Source Terms

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two kind of source terms are possible:

- volume source terms,

- surfaces source terms

Forces

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- **surface forces**, represented with a tensor because dependes on the orientation,
- **volume forces**, are distributed in the volume such as gravity, electromagnetic ....

$$\Sigma F = f_{\text{SURFACE}} + f_{\text{VOLUME}}$$

- **surface forces**, represented with a tensor because dependes on the orientation,
- **volume forces**, are distributed in the volume such as gravity, electromagnetic ….

$$f_{\text{SURFACE}} = \oint_{\partial\Omega} \sigma\left(\mathbf{x},t\right)\,\mathbf{n}\,d\partial\Omega = \int_{\Omega} \frac{\partial\,\sigma\left(\mathbf{x},t\right)}{\partial\mathbf{x}}\,d\Omega = \int_{\Omega} \frac{\partial\,\sigma_{\mathrm{i\,j}}\left(\mathbf{x},t\right)}{\partial x_{\mathrm{i}}}\,d\Omega$$

- $\sigma_{\mathrm{i\,j}}$ stress tensor in the flow,
- $\mathbf{n}$ normal to the border of the domain Ω

Stress tensor (Stokesian Fluid)

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In order to have a well defined set of equations, we need to introduce a **model** of the relation between the **state of the fluid** and the **stress tensor**.

There are many ways to define this relation, the most common is related to the so-called **Stokesian fluids**, which is based on

- The **stress tensor** is a continuous function **only** of the **strain rate tensor**, which is defined as: $\dot{\epsilon}_{\mathrm{i\,j}} \triangleq \left(\frac{\partial u_{\mathrm{i}}}{\partial x_{\mathrm{j}}} + \frac{\partial u_{\mathrm{j}}}{\partial x_{\mathrm{i}}}\right)$, and various thermodynamic state functions,
- We assume that the fluid is **isotropic** and **homogeneous**,
- When the strain rate is 0, then the stress tensor is defined by the isotropic pressure: $\dot{\epsilon}_{\mathrm{i\,j}} = 0 \Rightarrow \sigma_{\mathrm{i\,j}} \triangleq p\,\delta_{\mathrm{i\,j}}$

Stress tensor (Newtonian Fluid)

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$$\sigma_{\mathrm{i\,j}}\left(\mathbf{x},t\right) = \mathcal{F}_{\mathrm{Stokes}}\left(p,\mu,\lambda,\frac{\partial u_{\mathrm{p}}}{\partial x_{\mathrm{q}}}\right)$$

- If we assume to use a Stokesian fluid model with a **linear relation** between **stress and rate of strain**, we obatian the so-called **NEWTONIAN FLUID**.

- In the modern generalization of **Hooke's law**, each component of the stress tensor is a linear combination of all components of the strain rate tensor, and allows to satisfy all the requirements of a **NEWTONIAN FLUID**.

$$\begin{aligned}\sigma_{\mathrm{i\,j}}\left(\mathbf{x},t\right) &= -p\,\delta_{\mathrm{i\,j}} + \tau_{\mathrm{i\,j}}\left(\mathbf{x},t\right)\\ &= -p\,\delta_{\mathrm{i\,j}} + c_{\mathrm{i\,j\,p\,q}}\frac{\partial u_{\mathrm{p}}}{\partial x_{\mathrm{q}}}\end{aligned}$$

Hook's Law

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- From **symmetry and thermodynamics** conditions it is possible to reduce the free parameters in the tensor c_{ijpq} from $81 = 3^4$ to 2.

The condition to be imposed are:

$$\begin{cases} c_{\mathrm{ijpq}} & = & c_{\mathrm{jipq}} \\ c_{\mathrm{ijpq}} & = & c_{\mathrm{ijqp}} \\ c_{\mathrm{ijpq}} & = & c_{\mathrm{pqij}} \\ c_{\mathrm{ijpq}} & \Rightarrow & \text{ISOTROPIC TENSOR} \end{cases}$$

- From **symmetry and thermodynamics** conditions it is possible to reduce the free parameters in the tensor c_{ijpq} from $81 = 3^4$ to 2.

The relation between the stress tensor and the and the strain rate can then be rearranged in the following way:

$$c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$

Where (λ, μ) are the so-called **LAME CONSTANTS**.

- From **symmetry and thermodynamics** conditions it is possible to reduce the free parameters in the tensor $c_{\mathrm{i\,j\,p\,q}}$ from $81 = 3^4$ to 2.

$$
\begin{aligned}
\tau_{\mathrm{i\,j}} &= \left(\lambda\delta_{\mathrm{i\,j}}\delta_{\mathrm{p\,q}} + \mu\left(\delta_{\mathrm{i\,p}}\delta_{\mathrm{j\,q}} + \delta_{\mathrm{i\,q}}\delta_{\mathrm{j\,p}}\right)\right)\frac{\partial u_{\mathrm{p}}}{\partial x_{\mathrm{q}}} \\
&= \lambda\delta_{\mathrm{i\,j}}\frac{\partial u_{\mathrm{q}}}{\partial x_{\mathrm{q}}} + \mu\left(\frac{\partial u_{\mathrm{i}}}{\partial x_{\mathrm{j}}} + \frac{\partial u_{\mathrm{j}}}{\partial x_{\mathrm{i}}}\right) \\
&= \lambda\delta_{\mathrm{i\,j}}\dot{\epsilon}_{\mathrm{q\,q}} + 2\mu\,\dot{\epsilon}_{\mathrm{i\,j}}
\end{aligned}
$$

- From **symmetry and thermodynamics** conditions it is possible to reduce the free parameters in the tensor $c_{\mathrm{i\,j\,p\,q}}$ from $81 = 3^4$ to 2.

$$\sigma_{\mathrm{i\,j}} = -p\,\delta_{\mathrm{i\,j}} + \lambda\delta_{\mathrm{i\,j}}\,\dot{\epsilon}_{\mathrm{q\,q}} + 2\,\mu\,\dot{\epsilon}_{\mathrm{i\,j}}$$

$$\sigma_{\mathrm{i\,j}} = -p\,\delta_{\mathrm{i\,j}} + \left(\eta - \tfrac{2}{3}\mu\right)\delta_{\mathrm{i\,j}}\,\dot{\epsilon}_{\mathrm{q\,q}} + 2\,\mu\,\dot{\epsilon}_{\mathrm{i\,j}}$$

The energy conservation equation states that the energy change in time can happen if some power is provided/removed from the system.

It is possible to divide the power sources in two main groups, the mechanical and the thermal.

- mechanical power due to forces acting on surfaces (stress),
- mechanical power due to forces acting on the volume (field forces),
- thermal power due a thermal flow through a surface,
- thermal exchange in the volume due to irradiation.

The energy conservation equation states that the energy change in time can happen if some power is provided/removed from the system.

It is possible to divide the power sources in two main groups, the mechanical and the thermal.

$$\begin{aligned}\sum \mathbf{W} &= \sum w^{\text{MECH}} + \sum w^{\text{THE}} \\ &= w^{\text{MECH}}_{\text{SURFACE}} + w^{\text{MECH}}_{\text{VOLUME}} + w^{\text{THE}}_{\text{SURFACE}} + w^{\text{THE}}_{\text{VOLUME}}\end{aligned}$$

Navier Stokes Equations

Assumptions made

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- continuum hypothesis,
- Newtonian fluid,
- inertial frame,
- no electromagnetic forces,
- no chemical reactions,
- no relativistic phenomena,
- the provided thermodynamics.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} &= 0 \\ \dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial \rho u_j u_i}{\partial x_i} &= \dfrac{\partial \left(\tau_{ij} - p\,\delta_{ij}\right)}{\partial x_i} + \rho\, f_j \\ \dfrac{\partial \rho e}{\partial t} + \dfrac{\partial \rho e u_i}{\partial x_i} &= \dfrac{\partial\, \left(\tau_{ij}\, u_j - q_i - p\, u_i\right)}{\partial x_i} + \rho\, \left(f_j\, u_j + Q^{R}\right) \end{cases}$$

$$\begin{cases} \mathcal{T}\left(\rho\left(\mathbf{x}, t\right), p\left(\mathbf{x}, t\right), E\left(\mathbf{x}, t\right)\right) = 0 \\ \tau_{ij} = \lambda \delta_{ij} \dfrac{\partial u_q}{\partial x_q} + \mu \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right); \qquad \lambda = \eta - \dfrac{2}{3}\mu \end{cases}$$

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- incompressible Navier-Stokes can be obtained from the *complete* Navier-Stokes equations by imposing the condition that the density must be constant $\rho =$ **CONST**,

- by imposing this condition:
  - the continuity equation (conservation of mass) became the codition of **null divergence of the velocity field**,
  - the **energy equation** became **decoupled** from the other equations,
  - the **thermodynamics disapper**, and the pressure is just a scalar field used to satisfy the null divergence on the velocity field.

Incompressible Navier Stokes Equations

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$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + u_i \dfrac{\partial \rho}{\partial x_i} + \rho \dfrac{\partial u_i}{\partial x_i} = 0 \\
\rho \dfrac{\partial u_j}{\partial t} + \rho \left(u_j \dfrac{\partial u_i}{\partial x_i} + u_i \dfrac{\partial u_j}{\partial x_i} \right) = \dfrac{\partial}{\partial x_i} \tau_{ij} - \dfrac{\partial}{\partial x_i} p \, \delta_{ij} + \rho f_j \\
\dfrac{\partial \rho e}{\partial t} + \dfrac{\partial \rho e u_i}{\partial x_i} = \dfrac{\partial \left(\tau_{ij} u_j - q_i - p u_i \right)}{\partial x_i} + \rho \left(f_j u_j + Q^R \right)
\end{cases}
$$

$$
\begin{cases}
\mathcal{T}(\rho(\mathbf{x},t), p(\mathbf{x},t), E(\mathbf{x},t)) = 0 \\
\frac{\partial}{\partial x_i} \tau_{ij} = \lambda \delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial u_q}{\partial x_q} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_i} + \mu \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_i}; \qquad \lambda = \eta - \frac{2}{3} \mu
\end{cases}
$$

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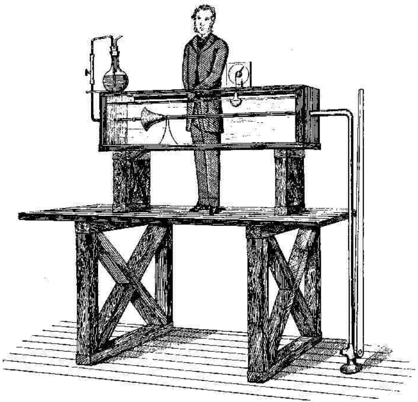
- In tensor notation:

$$\begin{cases} \dfrac{\partial u_i}{\partial x_i} = 0 \\ \rho \dfrac{\partial u_j}{\partial t} + \rho u_i \dfrac{\partial u_j}{\partial x_i} = -\dfrac{\partial p}{\partial x_j} + \mu \dfrac{\partial^2 u_j}{\partial x_i^2} + \rho f_j \end{cases}$$

- In vector notation:

$$\begin{cases} \nabla \mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\,\mathbf{u} = \dfrac{\mu}{\rho}\nabla^2 \mathbf{u} - \dfrac{1}{\rho}\nabla p \end{cases}$$

Thank you for your attention!



Where Reynold's number comes from

$$\begin{cases} \tilde{\nabla}\tilde{\mathbf{u}} & = & 0 \\ \dfrac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \left(\tilde{\mathbf{u}}\tilde{\nabla}\right)\tilde{\mathbf{u}} & = & \dfrac{1}{\mathrm{Re}}\tilde{\nabla}^2\tilde{\mathbf{u}} - \dfrac{1}{\rho}\tilde{\nabla}\tilde{p} \end{cases}$$

Energy budget

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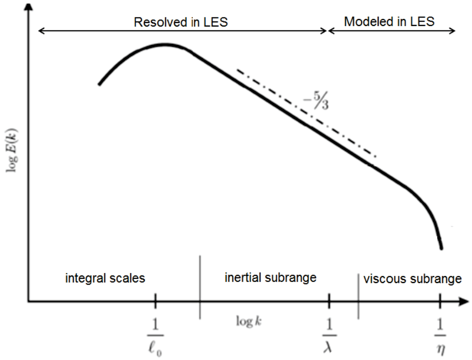

THE ENERGY CASCADE:

- the interaction between pressure gradients, inertial terms and viscous forces,
- Small vortices on the shoulder of bigger vortices, on the shoulders of bigger vortices …

No matter how powerful future computers will be, it will always exist a turbulence problem that can not be solved in human time![Maurizio Quadrio]

Many ways to compute the mean that lead to diffierent models (RANS,ILES,ELES)

$$x \triangleq \bar{x} + x', \quad \text{WITH} \quad \overline{(\bullet)} \rightarrow \text{MEAN OPERATOR} \tag{1}$$

The more intuitive is ILES, but concepts are the same also for other approaches

The closure problem

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- Businnesque hypothesis (algebraic) ν_t,
- 1 equation (differential) Spallart-Almaras,
- 2 equations (differential) $\kappa - \epsilon$,$\kappa - \omega$,
- Machine learning approaches?

Thank you for your attention!